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Monotone Approximation with Linear Differential Operators

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The subject of *monotone approximation* initiated in [2] has become a major trend in approximation theory. A typical problem in this subject is: given a positive integer k, approximate a given function whose kth derivative is $\geqslant 0$ by polynomials having this propery.

Here we generalize this problem by replacing the kth derivative with a linear differential operator of order k.

THEOREM: *Let h, k, p be integers, $0 \leqslant h \leqslant k \leqslant p$ and let f be a real function, $f^{(p)}$ continuous in $[-1, 1]$ with modulus of continuity $\omega(f^{(p)}, x)$ there. Let $a_j(x)$, $j = h, h+1, \dots, k$ be real functions, defined and bounded on $[-1, 1]$ and assume $a_h(x)$ is either $\geqslant$ some number $\alpha > 0$ or $\leqslant$ some number $\beta < 0$ throughout $[-1, 1]$. Consider the operator*

$$L = \sum_{j=h}^{k} a_j(x) \, [d^j/dx^j]$$

and suppose, throughout $[-1, 1]$,

$$L(f) \geqslant 0. \tag{1}$$

Then, for every integer $n \geqslant 1$, there is a real polynomial $Q_n(x)$ of degree $\leqslant n$ such that

$$L(Q_n) \geqslant 0 \qquad \text{throughout } [-1, 1]$$

and

$$\max_{-1 \leqslant x \leqslant 1} |f(x) - Q_n(x)| \leqslant C n^{k-p} \omega(f^{(p)}, 1/n)$$

where C is independent of n or f.

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Proof. Let n be an integer $\geqslant 1$. By a theorem of Trigub [4, 3], given a real function g, with $g^{(p)}$ continuous in $[-1, 1]$, there is a real polynomial $q_n(x)$ of degree $\leqslant n$ such that

$$\max_{-1 \leqslant x \leqslant 1} |g^{(j)}(x) - q_n^{(j)}(x)| \leqslant R_p n^{j-p} \omega(g^{(p)}, 1/n), \qquad j = 0, 1, ..., p, \quad (2)$$

where R_p is independent of n or g. Set

$$s_j \equiv \sup_{-1 \leqslant x \leqslant 1} |a_h^{-1}(x)\, a_j(x)|,$$

$$\eta_n = R_p \omega(f^{(p)}, 1/n) \sum_{j=h}^{k} s_j n^{j-p}.$$

I. Suppose, throughout $[-1, 1]$, $a_h(x) \geqslant \alpha > 0$. Let $Q_n(x)$ be a real polynomial of degree $\leqslant n$ so that

$$\max_{-1 \leqslant x \leqslant 1} |(f(x) + \eta_n (h!)^{-1} x^h)^{(j)} - Q_n^{(j)}(x)|$$

$$\leqslant R_p n^{j-p} \omega(f^{(p)}, 1/n), \qquad j = 0, 1, ..., p.$$

Then

$$\begin{aligned} \max_{-1 \leqslant x \leqslant 1} |f(x) - Q_n(x)| &\leqslant \eta_n (h!)^{-1} + R_p n^{-p} \omega(f^{(p)}, 1/n) \\ &\leqslant R_p \Big(1 + (h!)^{-1} \sum_{j=h}^{k} s_j\Big) n^{k-p} \omega(f^{(p)}, 1/n). \end{aligned} \quad (3)$$

Also if $-1 \leqslant x \leqslant 1$, then

$$\begin{aligned} a_h^{-1}(x)\, L(Q_n(x)) &= a_h^{-1}(x)\, L(f(x)) + \eta_n \\ &\quad + \sum_{j=h}^{k} a_h^{-1}(x)\, a_j(x) [Q_n(x) - f(x) - \eta_n (h!)^{-1} x^h]^{(j)} \\ &\geqslant \eta_n - \sum_{j=h}^{k} s_j R_p n^{j-p} \omega(f^{(p)}, 1/n) = 0 \end{aligned}$$

and hence $L(Q_n(x)) \geqslant 0$.

II. Suppose, throughout $[-1, 1]$, $a_h(x) \leqslant \beta < 0$. In this case let $Q_n(x)$ be a real polynomial of degree $\leqslant n$ such that

$$\max_{-1 \leqslant x \leqslant 1} |(f(x) - \eta_n (h!)^{-1} x^h)^{(j)} - Q_n^{(j)}(x)|$$

$$\leqslant R_p n^{j-p} \omega(f^{(p)}, 1/n), \qquad j = 0, 1, ..., p.$$

Again (3). Also if $-1 \leqslant x \leqslant 1$, then

$$a_h^{-1}(x)\,L(Q_n(x)) = a_h^{-1}(x)\,L(f(x)) - \eta_n + \sum_{j=h}^{k} a_h^{-1}(x)\,a_j(x)[Q_n(x) - f(x) + \eta_n(h!)^{-1}x^h]^{(j)}$$

$$\leqslant -\eta_n + \sum_{j=h}^{k} s_j R_p n^{j-p} \omega(f^{(p)}, 1/n) = 0$$

and hence $L(Q_n(x)) \geqslant 0$.

Remark. Suppose $a_h(x),\ldots, a_k(x)$ are continuous in $[-1, 1]$ and (1) is replaced by $L(f) > 0$. Disregard the assumption made in the Theorem on $a_h(x)$. For $n = 1, 2,\ldots,$ let $Q_n(x)$ be $q_n(x)$ of (2) for $g = f$. Then $Q_n(x)$ converges to f at the Jackson rate [1, p. 18, Theorem VIII] and at the same time, since $L(Q_n)$ converges uniformly to $L(f)$ on $[-1, 1]$, $L(Q_n) > 0$ throughout $[-1, 1]$ for all n sufficiently large.

References

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